Student Reasoning about Linear Algebra in Quantum Physics

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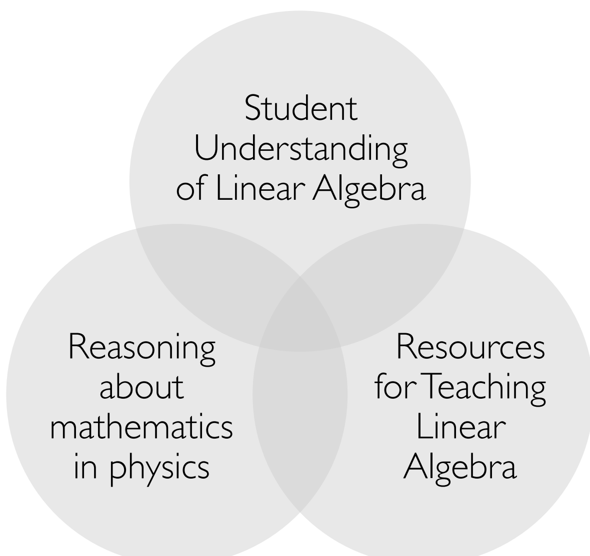
VIRGINIA TECH

My Research Interests

- Characterizing student understanding of undergraduate linear algebra
- Developing inquiry-oriented curriculum and instructor support materials for linear algebra [I O L A] **http://iola.math.vt.edu**
- Investigating students' reasoning about and use of mathematics in physics



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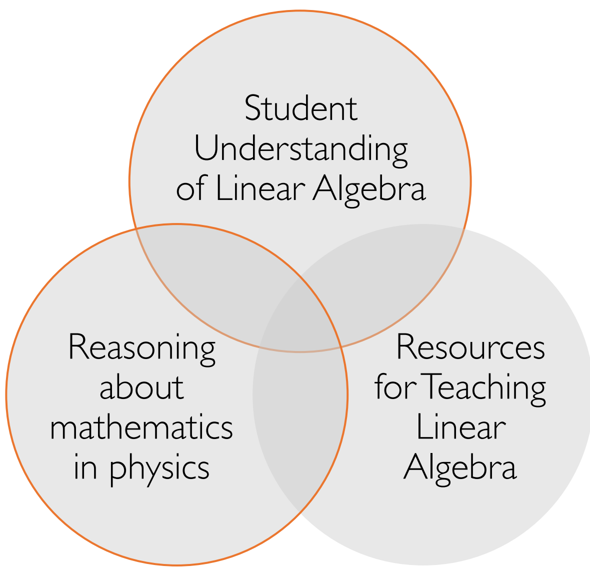

Goals for today's talk:

- Give an brief overview of Project LinAL-P
- Share results from two ongoing research threads of Project LinAL-P
  - Students' meta-representational competence (MRC) that is expressed while solving quantum mechanics problems that involve eigentheory
  - Student reasoning about linear combinations of eigenvectors

Project LinAL-P

Overarching Research Questions:

1. What are the various ways in which students reason about and symbolize concepts related to eigentheory in quantum physics?
2. How do students' language and symbols for concepts related to eigentheory compare and contrast across mathematics contexts and quantum physics contexts?
3. How might instructors' use of language and mathematical symbols relate to students' understanding, symbolization, and interpretation of eigentheory in quantum physics?

NSF Faculty Early Career Development Program, *CAREER: An Interdisciplinary Study of Learning: Student Understanding of Linear Algebra in Physics* (DUE 1452889), M. Wawro (PI); 2015-2020.

Project LinAL-P

Motivation from the DBER Report (2012) :

- Recommends "interdisciplinary studies of cross-cutting concepts and cognitive processes" (p. 3) in undergraduate STEM courses
- States that "gaps remain in the understanding of student learning in upper division courses" (p. 199)
- Posits that interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines … and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202)

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Project LinAL-P: Acknowledgments

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NSF Faculty Early Career Development Program, *CAREER: An Interdisciplinary Study of Learning: Student Understanding of Linear Algebra in Physics* (DUE 1452889), M. Wawro (PI); 2015-2020.

Why Linear Algebra and Eigentheory?

- Linear algebra is a key course across the STEM-related majors
- Eigenvectors and eigenvalues have widespread application in mathematics and beyond
  - Different equations, probability, graph theory, cryptography, matrix theory (stochastic processes, predator-prey models, connectivity of graphs, etc.)
  - (Quantum) physics, chemistry, economics, computer science, and engineering
- Understanding eigenvectors and eigenvalues builds upon understanding of multiple key ideas in linear algebra
  - Linear transformation, matrix operations, determinant, basis, solution set, linear (in)dependence, null space, subspace
  - Geometric and algebraic interpretations of these ideas
- Research has the potential to impact instruction in mathematics and other STEM fields

Part 1: Students' meta-representational competence (MRC)

- Physics students have to learn a new notation system and relate it to familiar notation systems as they learn quantum mechanics
- Physics students can exhibit flexibility in notation systems as they reason about ideas and solve problems that make use of linear algebra
- What relationships may exist between students' understanding of content (quantum mechanics or linear algebra) and their symbolization?

Wawro, M., Watson, K., & Christensen, W. (2017, February). *Meta-representation alcompetence with linear algebra in quantum mechanics*. Paper presented at the Research in Undergraduate Mathematics Education Conference.

Wawro, M., Watson, K., & Christensen, W. (2017, July). *Investigating Students' Meta-Representational Competence with Matrix Notation and Dirac Notation*. Paper presented at the Physics Education Research Conference, Cincinnati, OH.

Motivation

At the beginning of an interview with a student at the end of a Quantum Spins course:

Interviewer: So how do you feel like, using eigenvectors and eigenvalues, in Spins has been similar to and different from how you've experienced those in other classes?

A25: Uh, well, it's very similar because you're doing a lot of the same math…the difference especially in physics, you're looking at kets. In, in at first I was kind of jarring, like to- to try to do the math in kets. But now, it's kinda- it's kinda easier, there's problems, there certain problems…where there's two ways to do them, they're kind of parallel, you can do it and you can expand the- the state in- expand them as kets in a different basis, or you can write that state as as a, as a vector in that basis, and you can either do the matrix math for the like expectation values for example, **<u>you can do the matrix math or you can do the ket math</u>**, and sometimes it's, I'm finding that I, rather expand something in the ket.

$$\langle A \rangle = \langle \psi | \underset{S_x}{A} | \psi \rangle$$

$$= \left({}_x\langle + | \tfrac{-4}{5} + {}_x\langle - | \tfrac{-3i}{5} \right) \left(\tfrac{-\hbar 4}{10} |+\rangle_x + \left(\tfrac{-\hbar i 3}{10} \right) |-\rangle_x \right.$$

$$\frac{\hbar 16}{50} - \frac{9\hbar}{50} = \boxed{\frac{7\hbar}{50}}$$

$$\begin{pmatrix} -4/5 \\ i3/5 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & -\frac{i3}{5} \end{pmatrix}$$

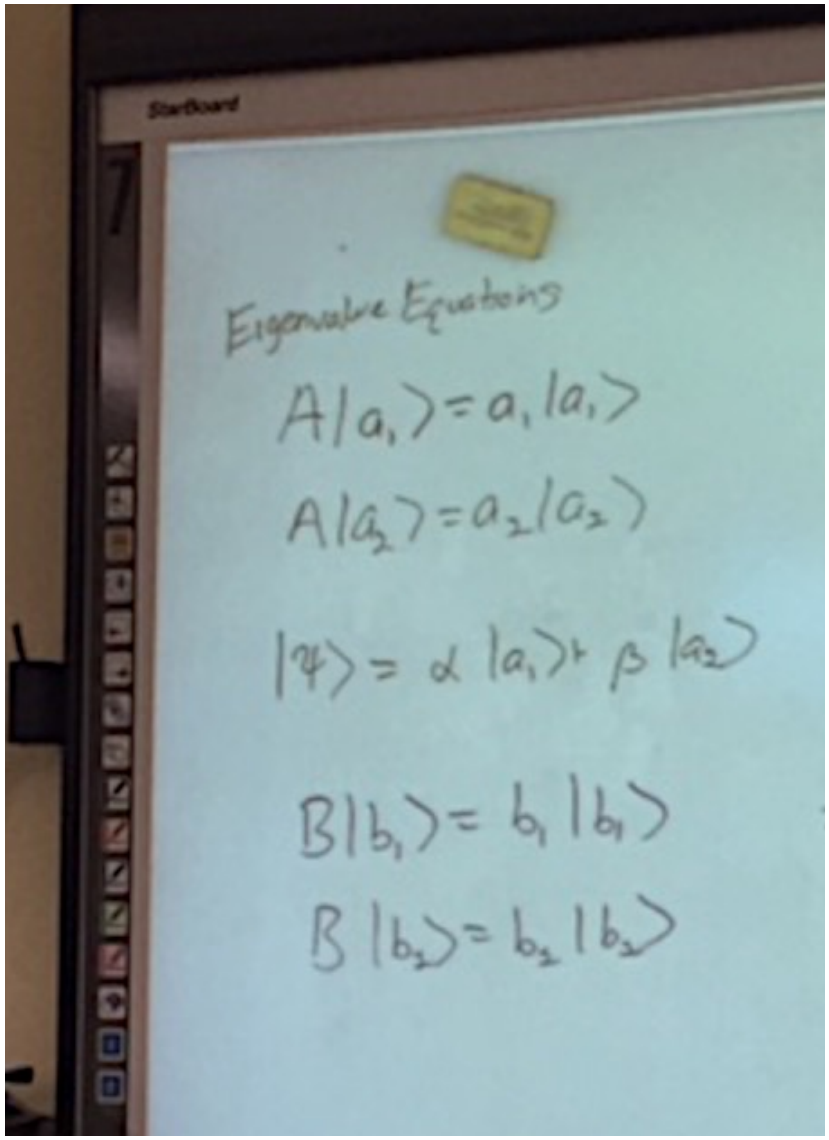
$$\begin{pmatrix} -\frac{4}{5} & \frac{-3i}{5} \end{pmatrix} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} -4/5 \\ i3/5 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{4}{5} & \frac{-3i}{5} \end{pmatrix} \begin{pmatrix} \frac{-4\hbar}{10} \\ \frac{-i3\hbar}{10} \end{pmatrix} = \frac{16\hbar}{50} - \frac{9\hbar}{50} = \boxed{\frac{7\hbar}{50}}$$

Comparing definitions and notation

<table>
<tr><th></th><th>Linear Algebra and its Applications (Lay, 2012)</th><th>Linear Algebra (Friedberg, Insel, & Spence, 2003)</th><th>Introduction to Quantum Mechanics (Griffiths, 2005)</th><th>Quantum Mechanics (McIntyre, 2012)</th></tr>
<tr><td>Linear transformation</td><td>A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that
(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and
(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.
(p. 204)</td><td>Let V and W be vector spaces (over F). We call a function $\mathrm{T}: \mathrm{V} \to \mathrm{W}$ a linear transformation from V to W if, for all $x, y \in \mathrm{V}$ and $c \in F$, we have:
(a) $\mathrm{T}(x + y) = \mathrm{T}(x) + \mathrm{T}(y)$ and
(b) $\mathrm{T}(cx) = c\mathrm{T}(x)$.
(p. 65)</td><td>A linear transformation $\hat{T}$ takes each vector in a vector space and “transforms” it into some other vector $\left(|\alpha\rangle \to |\alpha'\rangle = \hat{T}|\alpha\rangle\right)$, subject to the condition that the operation be linear:
$\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a\left(\hat{T}|\alpha\rangle\right) + b\left(\hat{T}|\beta\rangle\right)$,
for any vectors $|\alpha\rangle$, $|\beta\rangle$ and any scalars a, b.
(p. 441)</td><td rowspan="2">An operator is a mathematics object that acts or operates on a ket and transforms it into a new ket, for example $A|\psi\rangle = |\phi\rangle$. However, there are special kets that are not changed by the operation of a particular operator, except for a possible multiplicative constant, which we know does not change anything measurable about the state. An example of a ket that is not changed by an operator would be $A|\psi\rangle = a|\psi\rangle$. Such kets are known as eigenvectors of the operator A and the multiplicative constants are known as the eigenvalues of the operator.
(p. 34)</td></tr>
<tr><td>Eigen-vector and eigen-value</td><td>An eigenvector $\mathbf{x}$ of an n x n matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.
(p. 267)</td><td>Let T be a linear operator on vector space V. A nonzero vector $v \in \mathrm{V}$ is called an eigenvector of T if there exists a scalar λ such that $\mathrm{T}(v) = \lambda v$. The scalar λ is called the eigenvalue corresponding to the eigenvector v.
(p. 246)</td><td>In a complex vector space every linear transformation has “special vectors”…which are transformed into scalar multiples of themselves: $\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$; they are called eigenvectors of the transformation, and the (complex) number λ is their eigenvalue.
(p. 449).</td></tr>
</table>

Comparing definitions and notation



Example from student work

Consider the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Describe how you would determine the eigenvalues and eigenvectors of A.

$$A|\lambda\rangle = 2|\lambda\rangle$$

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$4a + 2b = 2a$$

$$a + 3b = 2b$$

$$a = -b$$

$$-4b + 2b = 2a$$

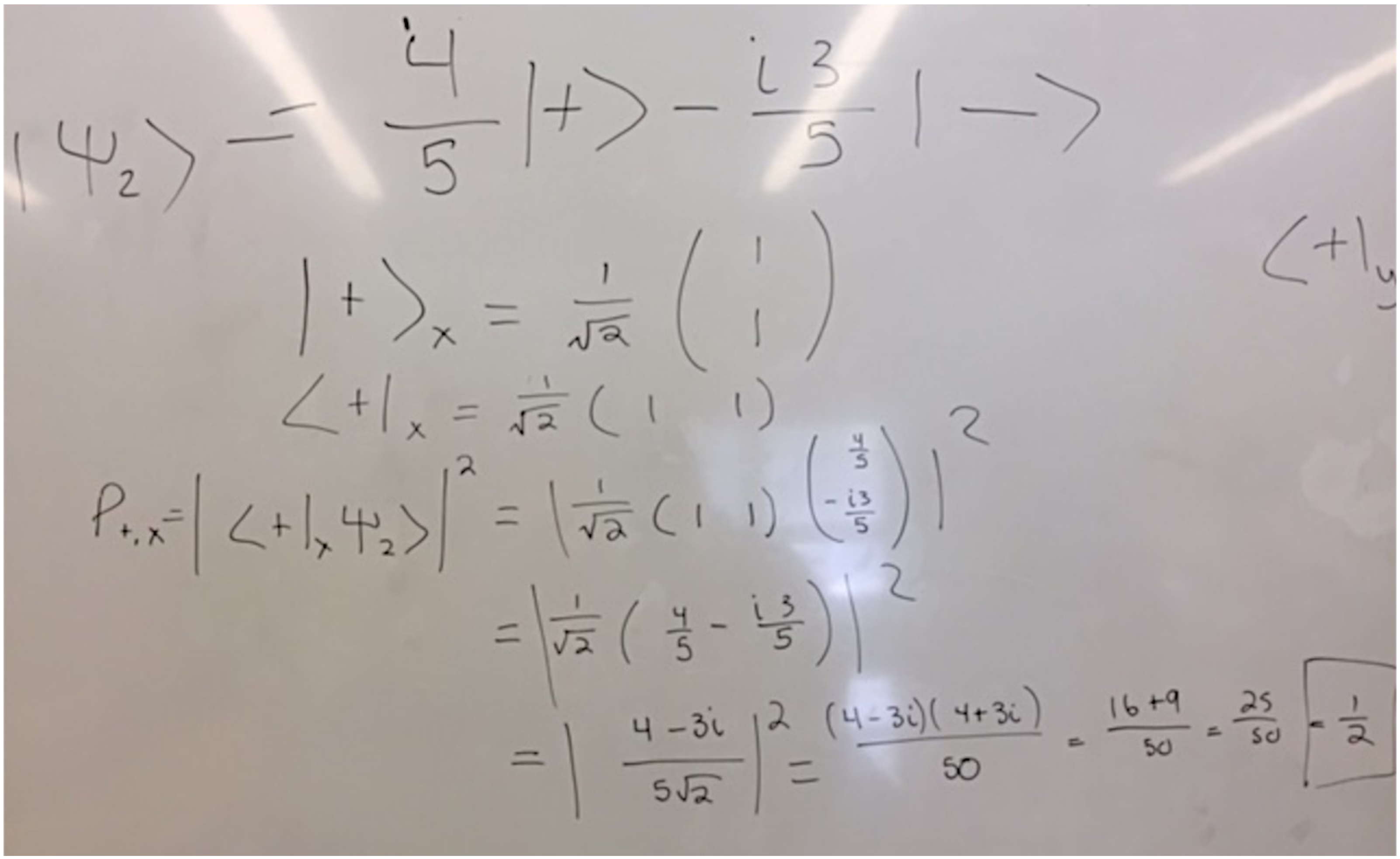
$$-2b = 2a$$

$$\boxed{-b = a}$$

$$|\lambda_1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example from student work

Consider the quantum state vector $|\Psi_2\rangle$. Calculate the probability that the spin component along the *x*-axis is measured to be spin up.



Student Understanding of Symbols and Representations

- diSessa (2002, 2004) examined students' developing competence with representations by defining what he calls Meta-Representational Competence (MRC).
- Meta-Representational Competence includes students' abilities to
  - Invent or design new representations
  - Critique and compare the adequacy of representations and judge their suitability for various tasks
  - Understand the purposes of representations generally and in particular contexts and understand how representations do the work they do for us
  - Explain representations
  - Learn new representations quickly and with minimal instruction

diSessa, A. A. (2002). *Symbolizing, modeling and tool use in mathematics education* (pp. 105–129).
diSessa, A. A. (2004) *Cognition and instruction,* 22(3), 293-331.

Research Question

What are some aspects of
Meta-Representational Competence
that exist in students' reflections on and
comparisons of matrix notation and Dirac
notation in quantum mechanics?

Student Understanding of Symbols and Representations

- Arcavi (1994, 2005) coined the term "symbol sense" to include the following ideas:
  - Being "friendly" with symbols
  - Reading through symbols
  - Engineering symbolic expressions
  - Understanding different meanings based on equivalent expressions
  - Choosing which aspects of a mathematical situation to symbolize
  - Using symbolic manipulations flexibly
  - Recognizing meaning within symbols at any step in the solution process
  - Sensing the different roles symbols can play in various contexts

Arcavi, A. (1994). *For the Learning of Mathematics, 14*(3), 24-35.
Arcavi, A. (2005). *For the Learning of Mathematics, 25*(2), 42-47.

Student Understanding of Symbols and Representations

- Gire and Price (2015) explored students' use of three different notations in Quantum Mechanics (Dirac, Matrix, and Wave Function)
- They identified four structural features of the notations for analysis:
  - Individuation, or "the degree to which important features are represented as separate and elemental" (p. 5)
  - Externalization, or "the degree to which elements and features are externalized with markings included in the representation" (p. 7)
  - Compactness
  - Symbolic Support for Computation

Gire, E., & Price, E. (2015). *Physical – Physics Education Research, 11*(2), 020109.

Methods – Data Collection

- Data was collected in a junior-level Quantum Spins class from a public research university in the northwestern United States

- The data sources include
  - Student interviews during their first week of class, before they started learning quantum mechanics
  - Video recordings of the class
  - Debriefing interviews with the instructor after class
  - Students' work from class and homework
  - Student interviews after their 3-wk, 7 hr/wk Quantum Spins course

Methods – Data Analysis

- Identified students who had engaged in the expectation value problem (4 of 8)

  Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system.
  Calculate the expectation value for the measurement of S_x.

- Read/watched their entire interview, identifying instances in which students explicitly talking about either notation
- Put relevant sections of the interview transcripts into one document and removed student identifiers
- Individually coded the sections, specifically attending to ideas related to MRC (diSessa, 2002, 2004) & the characteristics identified by Gire & Price (2015)
- Collectively shared our codes and decided upon a set for each transcript section
- Engaged in axial coding, categorizing our MRC codes by the characteristics of Dirac and matrix notations that students were attuned to

diSessa, A. (2002). *Symbolizing, modeling and tool use in mathematics education* (pp. 105–129).
Gire, E., & Price, E. (2015). *Physical – Physics Education Research, 11*(2), 020109.

MRC codes

A. Preference based on a value judgment

1. Clarity
2. Speed
3. Familiarity
4. "Likeability"
5. Ease of writing

B. Preference based on the problem or context

1. Useful in calculations
2. Makes more direct use of given relationships
3. Needs less information
4. Compactness
5. Individuation
6. Externalization

C. Understand the purposes of representations and how they do work for us

1. Has freedom to choose symbols
2. Has an ease with notation, writes symbols to mean what is personally desired
3. Aware of one's own progress in notation use
4. Able to "step back" and weigh options to decide which notation system is best

Jamie's work on the Expectation Value problem

$S_x = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$ $\quad \frac{1}{5}\begin{pmatrix} -4 \\ 3i \end{pmatrix}$

Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system.
Calculate the expectation value for the measurement of S_x.

$\langle S_x \rangle = \underline{\langle \psi | S_x | \psi \rangle}$

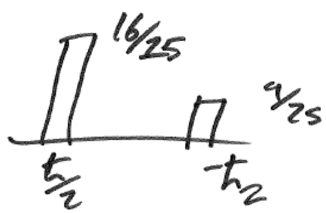
$S_x|\psi\rangle = S_x\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)$

$S_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle$

$= -\frac{4}{5}\frac{\hbar}{2}|+\rangle - \frac{i\hbar}{2}\frac{3}{5}|-\rangle$

$\langle \psi | S_x | \psi \rangle$

$= \frac{16}{25}\frac{\hbar}{2} - \frac{9}{25}\frac{\hbar}{2} = \frac{7}{25}\frac{\hbar}{2}$



$\frac{1}{5}\begin{pmatrix} -4 & -3i \end{pmatrix}\frac{1}{2}\begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}\begin{pmatrix} -4 \\ 3i \end{pmatrix}\frac{1}{5}$

$\frac{1}{50}\begin{pmatrix} -4 & -3i \end{pmatrix}\begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}\begin{pmatrix} -4 \\ 3i \end{pmatrix}$

$= \frac{1}{50}\begin{pmatrix} -4 & -3i \end{pmatrix}\begin{pmatrix} -4\hbar \\ -3i\hbar \end{pmatrix}$

$= \frac{1}{50}\left(16\hbar + 9(-1)\hbar\right)$

$= \frac{7\hbar}{50}$

$(S_z)_x = \begin{pmatrix} {}_x\langle + | S_z | + \rangle_x & \cdot + \cdot & - \\ - \cdot\ \cdot + \cdot & - & - \end{pmatrix}$

Jamie's work on the Expectation Value problem

So, this is very convenient because it's in the S_x basis. Um, so, basically all we need to do is put S_x in some matrix representation as — well, do we need to do matrix representation? I don't think we do. So, let's not worry about that. So, the expectation value of S_x is defined as this ψ, and the S_x, and ψ. And we're just going to include the ψ's in this — We're just going to drop the x subscript and just assume that it's, um, the x-basis….

When you first started, you almost started using matrix notation, and then you decided not to. Can you talk more about that?

So, ket — because you know this eigenvalue equation, ket notation just skips all that. Um, if you wanted to, you could have written this out as, uh, 1/5(-4 3i), But, you can see that's more confusing to go through — and then you have to look at S_x and, where you can write it as ħ/2, -ħ/2, 0, 0, and you have to do matrix multiplication. Um, actually, that might be quicker, honestly …
Which is, the exact same answer we got before, and it was substantially quicker. And in — I mean, I guess it depends how good you were with this kind of notation 'Cause, this is, I learned two and a half weeks ago, and this I learned almost a year ago.

Jamie's work on the Expectation Value problem

[C4] Able to "step back" and weigh options to decide which notation system is best

- Basically all we need to do is put Sx in some matrix representation as — well, do we need to do matrix representation? I don't think we do.
- if you wanted to, you could have written [C4] this out as, uh, 1/5[-4 3*i*]

[C2] Has an ease with notation, writes symbols to mean what is personally desired

- We're just going to drop the x subscript and just assume that it's, um, the x-basis.

[B2] Makes more direct use of given relationships

- Because you know this eigenvalue equation, ket notation just skips all that

[A1] Preference based on value judgment of clarity

- But, you can see that's more confusing to go through

[A2] Preference based on value judgment of speed

- Um, actually, that might be quicker, honestly

[A3] Preference based on value judgment of familiarity
[C3] Aware of one's own progress in notation use

- I guess it depends how good you were with this kind of notation 'cause, this is, I learned two and a half weeks ago, and this I learned almost a year ago.

Buzz's work on the Expectation Value Problem

Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system.
Calculate the expectation value for the measurement of S_x.

$$\langle A\rangle = \langle\psi|\underset{S_x}{A}|\psi\rangle$$

$$= \left({}_x\langle +|\frac{-4}{5} + {}_x\langle -|\frac{-3i}{5}\right)\left(-\frac{\hbar 4}{10}|+\rangle_x + \left(\frac{-\hbar i3}{10}|-\rangle_x\right.\right.$$

$$\frac{\hbar 16}{50} - \frac{9\hbar}{50} = \boxed{\frac{7\hbar}{50}}$$

500 480
~~48~~ 9

$\hbar/2$ $-\hbar/2$

$$10\frac{\hbar}{2} + 9\,\frac{-\hbar}{2}$$

p. 5

$$\begin{pmatrix}-4/5\\ i3/5\end{pmatrix}\begin{pmatrix}\frac{-4}{5} & -\frac{i3}{5}\end{pmatrix}$$

$$\begin{pmatrix}\frac{-4}{5} & \frac{-3i}{5}\end{pmatrix}\begin{pmatrix}\hbar/2 & 0\\ 0 & -\hbar/2\end{pmatrix}\begin{pmatrix}-4/5\\ i3/5\end{pmatrix}$$

$$\begin{pmatrix}\frac{-4}{5} & \frac{-3i}{5}\end{pmatrix}\begin{pmatrix}\frac{-4\hbar}{10}\\ \frac{-i3\hbar}{10}\end{pmatrix} = \frac{16\hbar}{50} - \frac{9\hbar}{50} = \boxed{\frac{7\hbar}{50}}$$

Buzz's work on the Expectation Value problem

Ok, so tell me, um, why you prefer this method, is it always or just this kind of problem, or tell me some more--

Uh…To be honest, I don't, I don't really, I don't really know why I prefer this, I- I think it's just because, um, <u>I like this notation</u>. This- this- this- specific notation like this- this to me is like a <u>cleaner way of writing </u>th- because that- that- I mean this and that I feel like are- are your starting points, so you, you start here with <u>this nice, like, looking thing, or you start here with this big array of numbers, and I- I- prefer this, even though you have to expand this into basically the same amount of information</u>. And also, the nice thing about, about this is, is it, actually this is really why it's- it's better is because you can, you can say ok S_x works- <u>acts directly on these kets, you can just get rid of the matrix altogether</u>, whereas this one's like, I mean you can do this with this one too, you can, but you have to recognize that, that you're in this basis and it's just, yeah it's not as, <u>it's not as nice</u> um, 'cause basically when you do this matrix multiplication, you have to do this matrix multiplication just to get where I got from here to here, which was literally I just said ok, looked at this and then I expanded it out because <u>I knew that this acting on that was gonna be ħ/2 and this acting on that was going to be -ħ/2, so I just put them over there, whereas here you kinda have to do that vector matrix thing first before you get those values</u>.

Buzz’s work on the Expectation Value problem

[A4] “Likeability”

- I don’t really know, uh, why I prefer this, I, I think it’s just because, um, I like the notation.

[A1] Clarity

- …this [pointing to ket notation] is like a clear way of writing…

[B4] Compactness

- …you start here with this nice, like, looking thing, or you start here with this big array of numbers, and I, I, I, I prefer this, even though you have to expand this into basically the same amount of information

[B2] Makes more direct use of given relationships

- …I just said ok, looked at this and then I expanded it out because I knew that this acting on that was gonna be h bar over 2 … whereas here kinda have to do that vector matrix thing first before you get those values.

[C4] Able to “step back” and weigh options to decide which notation system is best

- …there's a homework problem actually on the last homework where we were doing projections and um, it's actually easier in tho- in, in that, do the matrix multiplication…

[B6] Externalization

- …every time you write down a ket you're, you're very conscious of what basis you're in.

Additional examples of MRC from the data

“So, if like the problem's difficult, I do it out like explicitly in like matrix notation pretty much.” – C3

“I probably prefer doing things in matrix, just because it's simpler to write down. But Dirac I feel is more clear.” – C2

“…I think Dirac notation is helpful, where you can see the inner products happening… Visually, that's appealing” – C5

“I feel like Dirac notation is just like cheating. It's just like a short hand” – C3

“…there are times when I might just mess something up because it's matrix notation and…it's not as clear what everything is, or everything that's happening.” - C2

“…matrix notation explicitly shows you what you're doing” – C3

“I know Dirac notation through vector and matrix methods, so like I can't really separate the two.” – C6

“No one likes Dirac notation. Can I just like speak on that, like?” – C5

“…having a state vector, um, in this eigenvector states compacted into a single line is really, um, neat as in tidy.” – C6

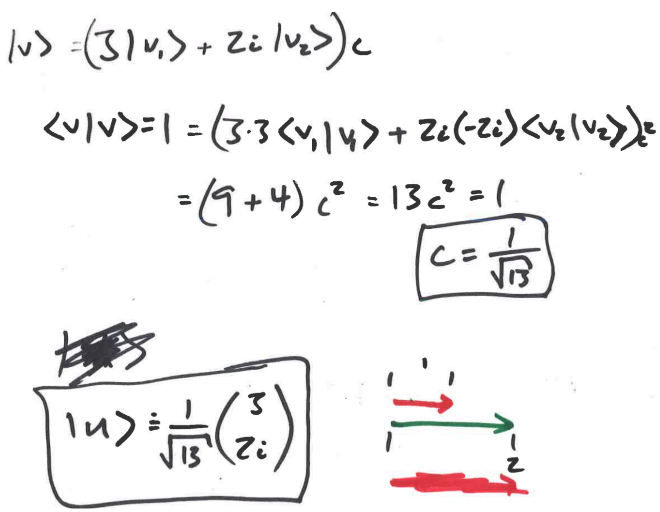
“Sometimes I feel more comfortable…with Dirac… because…I feel like it's easier to visualize what's going on.” – C2

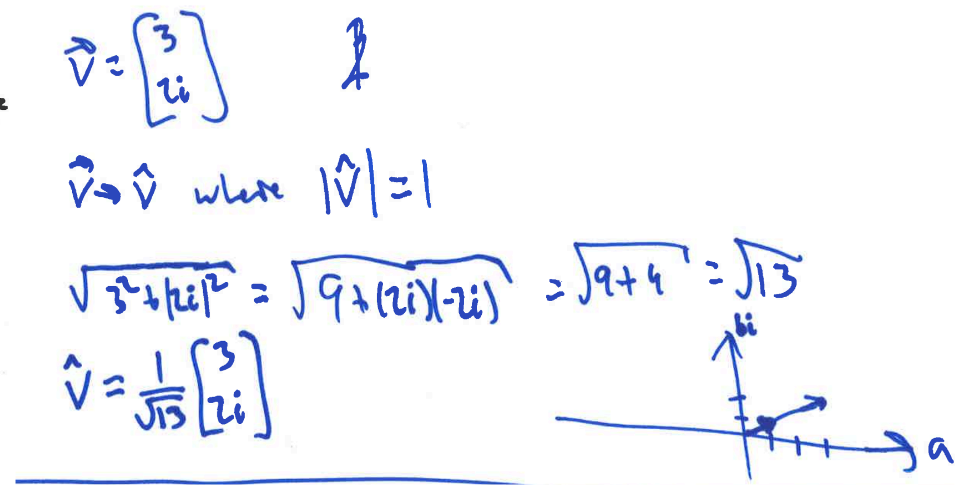
“…well I could write this out in Dirac notation, and maybe that makes it cleaner for like a little bit.” - C5

Next Steps

- Analyze student interviews from other data collection site, refine codes / categories as needed
  - Asked same Expectation Value problem
  - Were more explicit in designing prompts that might elicit MRC
- Investigate what relationships exist between students' understanding of content (quantum or linear algebra), their MRC, and notation system use in general

Read to student: "Normalize a vector whose components are 3 and 2*i*"



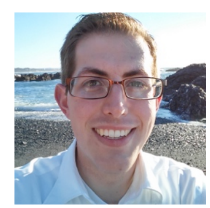


Part 2: Linear Combinations of Eigenvectors

How do students make sense of and reason about linear combinations of eigenvectors?

Wawro, M., Watson, K., & Zandieh, M. (2018, February). *Student Understanding of Linear Combinations of Eigenvectors.*. Paper presented at the Research in Undergraduate Mathematics Education Conference.

Wawro, M., Watson, K., & Zandieh, M. (accepted). Student understanding of linear combinations of eigenvectors. *ZDM- The International Journal on Mathematics Education*.



Student Understanding of Eigentheory

- Instructors often move back and forth between geometric, algebraic, and abstract modes of description and notation often without explicitly alerting students (Hillel, 2000)

- Students struggle to coordinate the two different mathematical processes captured in the equation $Ax = \lambda x$ to make sense of equality as "yielding the same result" (Thomas & Stewart, 2011)

nxn matrix vector in $\mathbb{R}^n$ scalar vector in $\mathbb{R}^n$

  - This interpretation of "equal" is nontrivial/novel to students (Henderson et al., 2010)

  - This may prevent students from making the needed symbolic progression from $Ax = \lambda x$ to $(A - \lambda I)x = 0$, which is needed to solve for eigenvalues and eigenvectors of a matrix A (Thomas & Stewart, 2011)

Student Understanding of Eigentheory

- The interpretation of "solution" in this setting, the <u>set of all vectors</u> x that make the equation true, entails a new level of complexity beyond solving equations such as $cx = d$ in $\mathbb{R}$ (Harel, 2000)

- Coordinating the number of eigenvectors corresponding to an eigenvalue with the dimension of the space spanned by the eigenvectors, and developing an understanding of eigenspace are both difficult for many students (Salgado & Trigueros, 2015)

- Calls have been made for further research into students' understanding of eigenspaces (Salgado & Trigueros, 2015)

Toward this end, we wanted to explore students' understanding of linear combinations of eigenvectors, specifically when the resultant vector is or is not an eigenvector itself

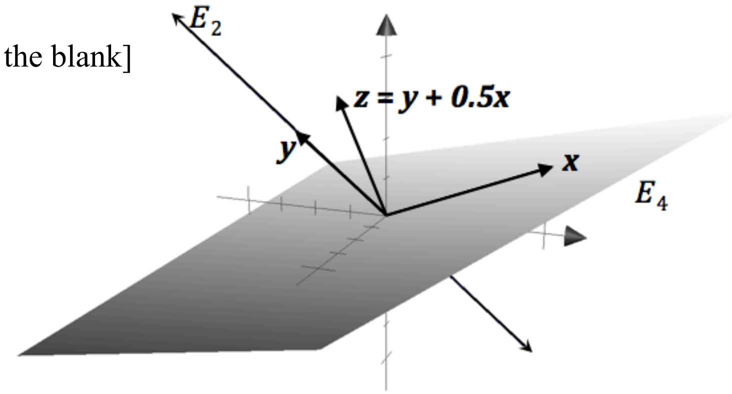
Methods – Data Collection

- Written responses to the 6-question Eigentheory Multiple-Choice Extended (MCE) Assessment Instrument (Watson, Wawro, Kerrigan, & Zandieh, 2017)
- Data from 5 classrooms:
  - Junior-level Quantum Spins class from a public research university in the northwestern US (32 students)
  - 3 sophomore-level introductory Linear Algebra classes, all with the same instructor, at a university in the mid-Atlantic US (27, 28, 29 students)
  - Senior-level Quantum Mechanics class from a public research university in the northeastern US (17 students)
- Focus on open-ended responses to Questions 3 and 5, which are about linear combinations of eigenvectors

Linear Combinations of Eigenvectors

3. Suppose A is a $n \times n$ matrix, and y and z are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $v = 5y + 5z$. Is $\boldsymbol{v}$ an eigenvector of A?
   - (a) Yes, v is an eigenvector of A with eigenvalue 2.
   - (b) Yes, v is an eigenvector of A with eigenvalue 5.
   - (c) No, v is not an eigenvector of A.

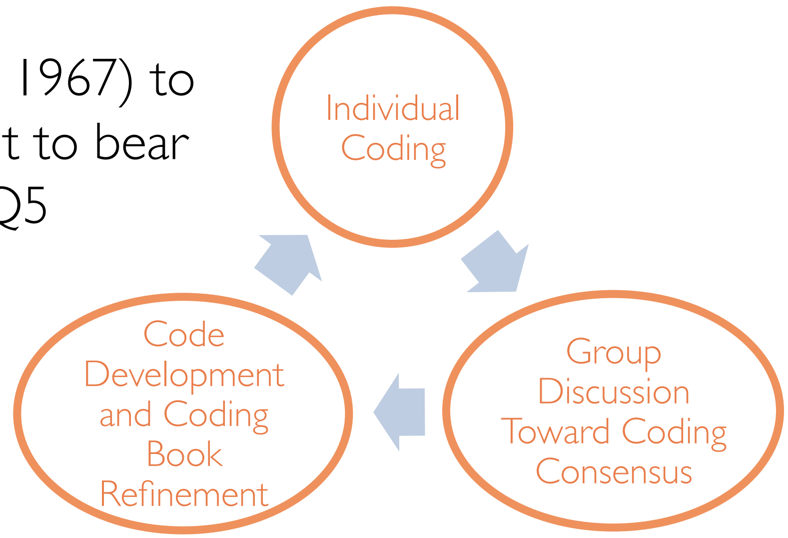
5. Suppose a 3x3 matrix B has two real eigenvalues: for eigenvalue 2 its eigenspace E_2 is one-dimensional, and for eigenvalue 4 its eigenspace E_4 is two-dimensional. Also suppose that vector $x \in \mathbb{R}^3$ lies on the plane created by the eigenspace E_4 and $y \in \mathbb{R}^3$ lies on the line created by the eigenspace E_2, as illustrated in the graph below. If $z = y + 0.5x$, which of the following is true?
   - (a) The vector z is an eigenvector of B with an eigenvalue of ______ [fill in the blank]
   - (b) The vector z is not an eigenvector of B.



Question 3	Question 5
Answer: Vector is an eigenvector	Answer: Vector is not an eigenvector
Vector is a LC of vectors from the same eigenspace	Vector is a LC of vectors from two different eigenspaces
No geometric/graphical imagery in the question wording	Geometric/graphical imagery in the question wording
Matrix is $n \times n$	Matrix is 3 x 3
Does not say "eigenspace" in its wording	Says "eigenspace" in its wording

Methods – Data Analysis (so far)

- Used Grounded Theory (Glaser & Strauss, 1967) to characterize the concepts students brought to bear as they justified their answers to Q3 and Q5



- The process resulted in 24 codes

$Ax=\lambda x$	$(A-\lambda I)x=0$	$\det(A-\lambda I)=0$	Calculation-based
Combination	E-vectors are LI	Vectors in LC are LI	Unique
Scale k	Scale λ	Geo-Vector	Geo-Transformation
Vector in e-space is also e-vector	Eigenspace	# e-vectors	Vectors in LC have same e-value
LC of e-vectors IS an e-vec	LC of e-vectors is NOT an e-vec	LC of e-vectors not necessarily e-vec	Only scalar multiples of e-vecs are e-vecs
Dimension	Geo – 'room' in the space / span	Size of the matrix	Miscellaneous

Selected Results

Two themes in student thinking:

1. Conflating scalars in a linear combination with eigenvalues
2. Reasoning about dimension of eigenspaces versus number of eigenvectors

1. Conflating the scalar in linear combo with the eigenvalue

3. Suppose A is a $n \times n$ matrix, and y and z are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $v = 5y + 5z$. Is $\boldsymbol{v}$ an eigenvector of A?
   (a) Yes, v is an eigenvector of A with eigenvalue 2.
   (b) Yes, v is an eigenvector of A with eigenvalue 5.
   (c) No, v is not an eigenvector of A.

Student responses coded with both Scale k and Scale λ

B70 chose (b):

$\lambda = 2$ $\quad Ay = 2y$

$Az = 2z$

$v = 5(y+z)$

5 is the scale factor of $x+y$ so $\lambda = 5$. Also, because y and z are eigenvectors of A, their combination is also an eigenvector of A.

1. Conflating the scalar in linear combo with the eigenvalue

3. Suppose A is a $n \times n$ matrix, and y and z are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $v = 5y + 5z$. Is $\boldsymbol{v}$ an eigenvector of A?
   (a) Yes, v is an eigenvector of A with eigenvalue 2.
   (b) Yes, v is an eigenvector of A with eigenvalue 5.
   (c) No, v is not an eigenvector of A.

Student responses coded with both Scale k and Scale λ

B81 chose (c): $\lambda = 2$

$V = 5y + 5z$

No, because an eigenvector ~~for~~ is defined as some linear combination defined by the eigenvalue, so that $Ax = \lambda x$, where x is the eigenvector and λ is the eigenvalue. The vectors y and z are being scaled by a factor of 5, and $\lambda = 2$ so they cannot be corresponding eigenvectors

1. Conflating the scalar in linear combo with the eigenvalue

Reminded us of final exam data (2014, southwestern university):

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

a. If you know that the line $k\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ (i.e., span$\left\{\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}\right\}$) is an eigenspace for this transformation, what is the stretch factor (eigenvalue) associated with this stretch direction. Show your work.

Answer: eigenvalue is -1

c. Circle the correct completion of each sentence, and fill in the blank if appropriate.

(i) The vector $\begin{bmatrix} 2 \\ -6 \\ 2 \end{bmatrix}$ is … an eigenvector of A with eigenvalue = _____ , or

not an eigenvector of A.

Answer: eigenvector with eigenvalue -1

	Given eigenvector for (c)				
	λ = -1	λ = 2	λ = -2	Other / Not Evec	Total
(a) Correct	13 (40.6%)	2	3	1	19
(a) Incorrect	0	9	0	4	13
Total	13	11 (34%)	3	5	32

2. Reasoning about dimension of eigenspaces versus # of eigenvectors

3. Suppose A is a $n \times n$ matrix, and y and z are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $v = 5y + 5z$. Is $\boldsymbol{v}$ an eigenvector of A?
   (a) Yes, v is an eigenvector of A with eigenvalue 2.
   (b) Yes, v is an eigenvector of A with eigenvalue 5.
   (c) No, v is not an eigenvector of A.

Student response coded with # e-vectors

B78 chose (c):

Technically, you could multiply the eigenvectors by any number and if you did so and another eigenvector was achieved there would be a possibility for infinite eigenvectors which doesn't make sense.

2. Reasoning about dimension of eigenspaces versus # of eigenvectors

5. Suppose a 3x3 matrix B has two real eigenvalues: for eigenvalue 2 its eigenspace E_2 is one-dimensional, and for eigenvalue 4 its eigenspace E_4 is two-dimensional. Also suppose that vector $x \in \mathbb{R}^3$ lies on the plane created by the eigenspace E_4 and $y \in \mathbb{R}^3$ lies on the line created by the eigenspace E_2, as illustrated in the graph below.
If $z = y + 0.5x$, which of the following is true?
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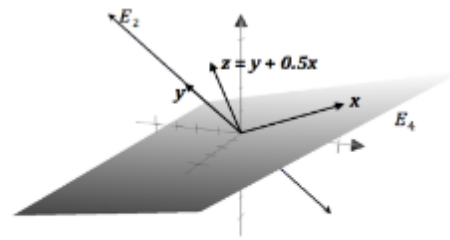


B59's response coded with # e-vectors and size of matrix

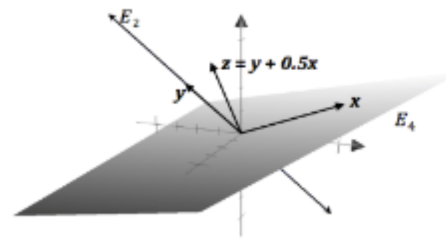
Matrix B already has 3 eigen vectors
so there's no room for a 4th

B66's response coded with linear combination , # e-vectors , and dimension

z is a linear combination of y & x, & there are already 3 eigenvectors for
3 dimensions, so z cannot be an eigenvector of B

2. Reasoning about dimension of eigenspaces versus # of eigenvectors

5. Suppose a 3x3 matrix B has two real eigenvalues: for eigenvalue 2 its eigenspace E_2 is one-dimensional, and for eigenvalue 4 its eigenspace E_4 is two-dimensional. Also suppose that vector $\vec{x} \in \mathbb{R}^3$ lies on the plane created by the eigenspace E_4 and $\vec{y} \in \mathbb{R}^3$ lies on the line created by the eigenspace E_2, as illustrated in the graph below. If $\vec{z} = \vec{y} + 0.5\vec{x}$, which of the following is true?
   (a) The vector $\vec{z}$ is an eigenvector of B with an eigenvalue of _____ [fill in the blank]
   (b) The vector $\vec{z}$ is not an eigenvector of B.



B58's response coded with eigenspace, size of matrix, dimension, span/room, and # e-vectors

in a 3x3 matrix there can only be 3 dimensions to the eigenspace. E_2 and E_4 together span the entire space of R^3 so there cannot be another eigenvector of B, besides E_2 and E_4.

B68's response coded with eigenspace, size of matrix, and dimension

B is nxn matrix where n=3. The dimensions of E_2 and E_4 add up to n, so there are no more eigenspaces. Because $\vec{z} \notin \{E_2, E_4\}$, $\vec{z}$ is not an eigenvector

Discussion and Next Steps

- Reasoning about scalars in linear combos of eigenvectors
  - Scalar(s) in the linear combination: independent of matrix operation
  - Eigenvalues are scalars: directly rely on matrix operation

- Reasoning about a finite number of eigenvectors
  - May be familiar from finding a basis for an eigenspace
  - May be familiar from solving diagonalization problem

Both require linearly independent eigenvectors

Next Steps

- Continue analysis of the open-ended data for additional findings
- Compare students across both questions for additional insights
- Investigate the structure of student responses by analyzing where/how the coded statements appeared (w/ Toulmin)

Thanks for Listening

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